Comment Math.
Univ. St. Pauli
XXVIII—2, 1979

On the unified presentation of operational formulas and generating functions for certain classical polynomials

by

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(Received March 24, 1979)

Abstract

The present paper attempts to present an elegant generalization of operational formulas and generating functions which have been very recently obtained by Patil and Thakore [5] for extended Jacobi polynomials $\{F_n(\alpha, \beta; x)\}$, and by Srivastava and Singhal [8] for generalized polynomials $\{T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r)\}$. The results obtained here include linear, bilateral and trilateral generating functions, and operational formulas for the polynomials $\{S_n^{(\alpha,\beta)}(x, a, b, c, d, e, f, p, r) \mid n=0, 1, 2, \cdots\}$ defined by (1.3) below.

1. Introduction and definition

Recently Patil and Thakare [5] have obtained some operational formulas and generating functions for the polynomials $\{F_n(\alpha, \beta; x)\}$ (first studied by Fujiwara [1]) defined by

$$
\begin{aligned}
F_n(\alpha, \beta; x) &= \frac{(-c)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta}D^n\{(x-a)^{n+\alpha}(b-x)^{n+\beta}\} \\
&= \left(\frac{-\lambda}{b-a}\right)^n \frac{(x-a)^{-\alpha}(b-x)^{-\beta}}{n!}D^n\{(x-a)^{n+\alpha}(b-x)^{n+\beta}\} ,
\end{aligned} \tag{1.1}
$$

where

$$\lambda = c(b-a) .$$

It may be observed that Srivastava and Singhal [8] have succeeded in obtaining more general results for the polynomials $[T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r)\}$ defined by

$$
\begin{aligned}
T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r) &= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n!} \\
&\quad \times \exp(px^r)\cdot D^n\{(ax+b)^{n+\alpha}(cx+d)^{n+\beta}\cdot \exp(-px^r)\}
\end{aligned} \tag{1.2}
$$

In the present paper we consider the polynomial system $\{S_n^{(\alpha,\beta)}(x, a, b, c, d, e, f, p, r)\}$ defined by

$$(1.3)\quad \begin{aligned} S_n^{(\alpha,\beta)}(x, a, b, c, d, e, f, p, r) &= \frac{(ax^e+b)^{-\alpha}(cx^f+d)^{-\beta}}{n!} \\ &\times \exp(px^r)\cdot D^n\{(ax^e+b)^{n+\alpha}(cx^f+d)^{n+\beta}\exp(-px^r)\}\ . \end{aligned}$$

In (1.3) the parameters e and f can take any complex values. Obviously polynomials defined by (1.2) can be obtained as a very particular case of (1.3) by putting $e=f=1$. In particular, we mention the following connections

$$(1.4)\quad S_n^{(\alpha,\beta)}(x, 0, b, 0, d, 1, 1, 1, 2) = \frac{(-bd)^n}{n!}\cdot H_n(x)$$

$$(1.5)\quad S_n^{(\alpha,\beta)}(x, a, 0, 0, d, 1, 1, 1, 1) = (ad)^n\cdot L_n^{(\alpha)}(x)\ .$$

$$(1.6)\quad \begin{aligned} S_n^{(\alpha,\beta)}(x, a, -a, c, c, 1, 1, 0, r) &= S_n^{(\alpha,\beta)}(x, a, -a, c, c, 1, 1, p, 0) \\ &= (2ca)^n\cdot P_n^{(\alpha,\beta)}(x) \end{aligned}$$

$$(1.7)\quad \begin{aligned} S_n^{(\alpha,\beta)}(x, 1, -a, -1, b, 1, 1, 0, r) &= S_n^{(\alpha,\beta)}(x, 1, -a, -1, b, 1, 1, p, 0) \\ &= \left(\frac{a-b}{\lambda}\right)^n\cdot F_n[\alpha, \beta; x]\ , \end{aligned}$$

where $F_n[\alpha, \beta; x]$ are the extended Jacobi polynomials defined by (1.1) above.

$$(1.8)\quad S_n^{(\alpha,\beta)}(x, a, b, c, d, 1, 1, p, r) = T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r)\ ,$$

where $T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r)$ are the Srivastava and Singhal polynomials defined by (1.2) above.

$$(1.9)\quad S_n^{(\alpha,0)}(x, a, 0, 0, 0, 1, 1, 2, -1) = \frac{(2ac)^n}{n!}\cdot Y_n^{(\alpha)}(x)\ ,$$

where $Y_n^{(\alpha)}(x)$ denotes the generalized Bessel polynomial of Krall and Frink [3] defined by

$$(1.10)\quad \begin{aligned} Y_n^{\alpha}(x) &= {}_2F_0\left[-n,\ n+\alpha+1;\ \text{——};\ -\frac{1}{2}x\right] \\ &= \frac{x^{-\alpha}}{2^n}\cdot\exp\left(\frac{2}{x}\right)\cdot D^n\left[x^{2n+\alpha}\cdot\exp\left(-\frac{2}{x}\right)\right] \end{aligned}$$

$$(1.11)\quad S_n^{(\alpha-n,\beta)}(x, a, 0, 0, d, 1, 1, p, r) = \frac{(-adx)^n}{n!} H_n^r(x, \alpha, p)$$

Here $H_n^r(x, \alpha, p)$ is the generalized Hermite polynomial defined by

$$(1.12)\quad H_n^r(x, \alpha, p) = (-1)^n\cdot x^{-a}\exp(px^r)\cdot D^n\{x^{\alpha}\cdot\exp(-px^r)\}$$

introduced earlier by Gould and Hopper [2].

2. Operational formulas

In this section we obtain some operational formulas for the polynomials defined by (1.3) by making use of the differential operator $\delta=x(d/dx)$ which possesses the following properties:

$$x^nD^n=\delta(\delta-1)\cdots(\delta-n+1)\,, \tag{2.1}$$

and

$$f(\delta)\cdot\exp\{g(x)\}\cdot h(x)=\exp\{g(x)\}\cdot f\{\delta+xg'(x)\}\cdot h(x)\,. \tag{2.2}$$

By assuming Y to be sufficiently differential function and using the properties (2.1) and (2.2), we get

$$\begin{aligned}&(ax^e+b)^{-\alpha}\cdot(cx^f+d)^{-\beta}\cdot\exp(px^r)\cdot D^n\{(ax^e+b)^{n+\alpha}(cx^f+d)^{n+\beta}\cdot\exp(-px^r)Y\}\\ &=\left\{\frac{(ax^e+b)(cx^f+d)}{x}\right\}^n\prod_{i=1}^{n}\Big[\delta+\frac{ae(n+\alpha)}{ax^e+b}\cdot x^e+\frac{cf(n+\beta)\cdot x^f}{cx^f+d}\\ &\quad -prx^r-i+1\Big]Y\end{aligned} \tag{2.3}$$

whereas, by using Leibniz' rule and (1.3) again, we observe that the left-hand-side of (2.3) can also be expressed in the form:

$$n!\sum_{k=0}^{n}\frac{(ax^e+b)^k(cx^f+d)^k}{k!}\cdot S_{n-k}^{(\alpha+k,\beta+k)}(x,\,a,\,b,\,c,\,d,\,e,\,f,\,p,\,r)D^kY\,.$$

Equivalence of the two expressions yields the following operational formula.

$$\begin{aligned}&\prod_{i=1}^{n}\left[\delta+\frac{ae(n+\alpha)}{ax^e+b}x^e+\frac{cf(n+\beta)}{cx^f+d}x^f-prx^r-i+1\right]Y\\ &\quad=n!\left\{\frac{x}{(ax^e+b)(cx^f+d)}\right\}^n\sum_{k=0}^{n}\frac{(ax^e+b)^k\cdot(cx^f+d)^k}{k!}\\ &\quad\times S_{n-k}^{(\alpha+k,\beta+k)})(x,\,a,\,b,\,c,\,d,\,e,\,f,\,p,\,r)D^kY\,.\end{aligned} \tag{2.4}$$

For $Y=1$, (2.3) gives the following operational formula

$$\begin{aligned}&\prod_{i=1}^{n}\left[\delta+\frac{ae(n+\alpha)}{ax^e+b}\cdot x^e+\frac{cf\cdot(n+\beta)x^f}{cx^f+d}-prx^r-i+1\right]\cdot 1\\ &\quad=n!\left\{\frac{x}{(ax^e+b)(cx^f+d)}\right\}^nS_n^{(\alpha,\beta)}(x,\,a,\,b,\,c,\,d,\,e,\,f,\,p,\,r)\,.\end{aligned} \tag{2.5}$$

Now in (1.3) we replace n by $n+m$ to get

$$\begin{aligned}S_{n+m}^{(\alpha,\beta)}(x,\,a,\,b,\,c,\,d,\,e,\,f,\,p,\,r)&=\frac{(ax^e+b)^{-\alpha}(cx^f+d)^{-\beta}}{(n+m)!}\\ &\quad\times\exp(px^r)D^{m+n}[(ax^e+b)^{n+m+\alpha}(cx+d)^{n+m+\beta}\exp(-px^r)]\\ &=\frac{m!}{(n+m)!}(ax^e+b)^{-\alpha}\cdot(cx^f+d)^{-\beta}\cdot\exp(px^r)\\ &\quad\times D^n[(ax^e+b)^{n+\alpha}\cdot(cx^f+d)^{n+\beta}\cdot\exp(-px^r)\\ &\quad S_m^{(\alpha+n,\beta+n)}(x,\,a,\,b,\,c,\,d,\,e,\,f,\,p,\,r)\,.\end{aligned}$$

By using Leibnitz rule this reduces to the following form

$$\binom{n+m}{n} S_{n+m}^{(\alpha,\beta)}(x,\ a,\ b,\ c,\ d,\ e,\ f,\ p,\ r)$$

$$(2.6)\qquad =\sum_{k=0}^{n} \frac{(ax^e+b)^k\cdot(cx^f+d)^k}{k!} S_{n-k}^{(\alpha+k,\beta+k)}(x,\ a,\ b,\ c,\ d,\ e,\ f,\ p,\ r) \times D^k S_m^{(\alpha+n,\beta+n)}(x,\ a,\ b,\ c,\ d,\ e,\ f,\ p,\ r)\ .$$

3. Papticular cases

In this section we show how the operational formulas (2.4) and (2.5) can be used to obtain as particular cases the operational formulas for Hermite, Laguerre, Jacobi, extended Jacobi, Bessel and Srivastava and Singhal generalized polynomials without using any limiting processes.

(I) *Hermite Polynomials.* Applying (2.4) and (2.5) to (1.4), we get the following operational formulas for Hermite polynomials

$$(3.1)\qquad \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} H_{n-k}(x) D^k Y = \prod_{i=1}^{n} [\delta-2x^2-i+1] Y\ ,$$

and

$$(3.2)\qquad (-x)^n H_n(x) = \prod_{i=1}^{n} [\delta-2x^2-i+1]\cdot 1\ .$$

(II) *Laguerre Polynomials.* Comparing (2.4) and (2.5) with (1.5) we obtain the following formulas for Laguerre polynomials.

$$(3.3)\qquad \sum_{k=0}^{n} \frac{n!}{k!}\cdot x^k L_{n-k}^{\alpha+k}(x) D^k Y = \prod_{i=1}^{n} [\delta+n+\alpha-x-i+1] Y\ ,$$

and

$$(3.4)\qquad n!\, L_n^{(\alpha)}(x) = \prod_{i=1}^{n} [\delta+n+\alpha-x-i+1]\cdot 1\ .$$

(III) *Jacobi Polynomials.* From (1.6), (2.4) and (2.5), we get

$$(3.5)\qquad \sum_{k=0}^{n} \frac{n!}{k!} \left(\frac{-2}{1-x^2}\right)^{n-k} \cdot x^n \cdot P_{n-k}^{(\alpha+k,\beta+k)}(x) D^k Y \\ \prod_{i=1}^{n} \left[\delta - \frac{(n+\alpha)x}{1-x} + \frac{(n+\beta)x}{x+1} - i+1\right] Y\ ,$$

and

$$(3.6)\qquad n! \left(\frac{-2x}{1-x^2}\right)^n P_n^{(\alpha,\beta)}(x) = \prod_{i=1}^{n} \left[\delta - \frac{(n+\alpha)}{1-x}x + \frac{(n+\beta)}{x+1}x - i+1\right]\cdot 1\ .$$

(IV) *Extended Jacobi Polynomials.* (2.4) and (2.5) when applied to (1.7) yield the following

$$
(3.7)\quad \begin{aligned} &n!\left[\frac{(a-b)x}{\lambda(x-a)(b-x)}\right]^n \sum_{k=0}^{n} \frac{1}{k!}\left\{\frac{\lambda}{(a-b)}(x-a)(b-x)\right\}^k \\ &\cdot F_{n-k}(\alpha+k,\ \beta+k;\ x)D^k Y \\ &=\prod_{i=1}^{n}\left[\delta+\frac{(n+\alpha)x}{x-a}-\frac{(n+\beta)x}{b-x}-i+1\right]Y\ , \end{aligned}
$$

and

$$
(3.8)\quad \begin{aligned} &n!\left[\frac{(a-b)x}{\lambda(x-a)(b-x)}\right]^n F_n(\alpha,\ \beta;\ x) \\ &=\prod_{i=1}^{n}\left[\delta+\frac{(n+\alpha)x}{x-a}-\frac{(n+\beta)x}{b-x}-j+1\right]1\ . \end{aligned}
$$

Note. (3.7) and (3.8) are the main operational formulas of Patil and Thakare [5]. Also on comparing (2.6) with (1.7) we obtain the following formnla of the same authors.

$$
(3.9)\quad \begin{aligned} &\binom{m+n}{n} F_{n+m}(\alpha,\ \beta;\ x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\left(\frac{\lambda}{b-a}\right)^k (x-a)^k(b-x)^k \\ &\times F_{n-k}(\alpha+k,\ \beta+k;\ x)\cdot D^k F_m(\alpha+n,\ \beta+n;\ x) \end{aligned}
$$

(V) *Srivastava' Polynomials* $T_n^{(\alpha,\beta)}(x,\ a,\ b,\ c,\ d,\ p,\ r)$. (2.4) and (2.5) when applied to (1.8), gives us

$$
(3.10)\quad \begin{aligned} &\prod_{i=1}^{n}\left[\delta+\frac{a(n+\alpha)x}{ax+b}+\frac{c(n+\beta)x}{cx+d}-prx^r-i+1\right]Y \\ &=n!\left\{\frac{x}{(ax+b)(cx+d)}\right\}^n \sum_{k=0}^{n}\frac{(ax+b)^k(cx+d)^k}{k!} \\ &\times T_{n-k}^{(\alpha+k,\beta+k)}(x,\ a,\ b,\ c,\ d,\ p,\ r)D^k Y\ , \end{aligned}
$$

and

$$
(3.11)\quad \begin{aligned} &\prod_{i=1}^{n}\left[\delta+\frac{a(n+\alpha)}{ax+b}\cdot x+\frac{c(n+\beta)\cdot x}{cx+d}-prx^r-i+1\right]\cdot 1 \\ &=n!\left\{\frac{x}{(ax+b)(cx+d)}\right\}^n T_n^{(\alpha,\beta)}(x,\ a,\ b,\ c,\ d,\ p,\ r) \end{aligned}
$$

Note. (3.10) and (3.11) are the main operational formulas of Srivastava and Singhal [8]. Also on comparing (3.6) with (1.8) we obtain the following formula of the same authors.

$$
(3.12)\quad \begin{aligned} &\binom{m+n}{n} T_{n+m}^{(\alpha,\beta)}(x,\ a,\ b,\ c,\ d,\ p,\ r) \\ &=\sum_{k=0}^{n}\frac{(ax+b)^k(cx+d)^k}{k!}T_{n-k}^{(\alpha+k,\beta+k)}(x,\ a,\ b,\ c,\ d,\ p,\ r) \\ &\times D^k T_m^{(\alpha+n,\beta+n)}(x,\ a,\ b,\ c,\ d,\ p,\ r)\ . \end{aligned}
$$

(VI) *Generalized Bessel Polynomials of Krall and Frink.* Comparing (2.4) and (2.5) with (1.9), we at once arrive at the following results:

$$\prod_{i=1}^{n}\left[\delta+2n+\alpha+\frac{2}{x}-i+1\right]\cdot Y$$

(3.13)
$$=\sum_{k=0}^{n}\binom{n}{k}x^{-n+2k}\cdot 2^{n-k}\cdot Y_{n-k}^{(\alpha+2k)}(x)D^{k}\cdot Y\ ,$$

and

(3.14)
$$\prod_{i=1}^{n}\left[\delta+2n+\alpha+\frac{2}{x}-i+1\right]\Big|=\left(\frac{2}{x}\right)^{n}\cdot Y_{n}^{\alpha}(x)\ .$$

4. Generating functions

In this section, we prove the following lemma:

LEMMA. *Let* $S_n^{(\alpha,\beta)}(x, a, b, c, d, e, f, p, r)$ *be defined as by* (1.3), *then*

(4.1)
$$\sum_{n=0}^{\infty}\binom{m+n}{n}S_{m+n}^{(\alpha-n,\beta-n)}(x, a, b, c, d, e, f, p, r)t^{n}$$
$$=\left[\frac{a\{x+(ax^{e}+b)(cx^{f}+d)t\}^{e}+b}{ax^{e}+b}\right]^{\alpha}\left[\frac{c\{x+(ax^{e}+b)(cx^{f}+d)t\}^{f}+d}{cx^{f}+d}\right]^{\beta}$$
$$\times\exp\{px^{r}-p(x+(ax^{e}+b)(cx^{f}+d)t)^{r}\}$$
$$\times S_{m}^{(\alpha,\beta)}(x+(ax^{e}+b)(cx^{f}+d)t, a, b, c, d, e, f, p, r)\ .$$

Proof. By making use of the definition (1.3), we observe that

$$\sum_{n=0}^{\infty}\binom{m+n}{n}S_{m+n}^{(\alpha-n,\beta-n)}(x, a, b, c, d, e, f, p, r)\left(\frac{t}{(ax^{e}+b)(cx^{f}+d)}\right)^{n}$$
$$=(ax^{e}+b)^{-\alpha}(cx^{f}+d)^{-\beta}\cdot\exp(px^{r})\frac{1}{m!}D^{m}\left[\sum_{n=0}^{\infty}\frac{1}{n!}D^{n}\{(ax^{e}+b)^{m+\alpha}\right.$$
$$\left.\times(cx^{f}+d)^{m+\beta}\}\exp(-px^{r})t^{n}\right],$$

which on applying the Lagranges theorem [6]

(4.2)
$$\frac{f(y)}{1-t\varphi'(y)}=\sum_{n=0}^{\infty}\frac{t^{n}}{n!}\cdot D^{n}\{[\varphi(x)]^{n}f(x)\},\ y=x+t\varphi(y)$$

yields,

$$\sum_{n=0}^{\infty}\binom{m+n}{n}S_{m+n}^{(\alpha-n,\beta-n)}(x, a, b, c, d, e, f, p, r)\left(\frac{t}{(ax^{e}+b)(cx^{f}+d)}\right)^{n}$$
$$=\frac{(ax^{e}+b)^{-\alpha}(cx^{f}+d)^{-\beta}}{m!}\cdot\exp(px^{r})\left(\frac{d}{dy}\right)^{m}$$
$$\times[(ay^{e}+b)^{m+\alpha}(cy^{f}+d)^{m+\beta}\cdot\exp(-py^{r})]\ ,$$

where $y=x+t$.

Now, in view of the definition (1.3), the above equation reduces to the desired form.

It is interesting to note that (4.1) can be used to obtain a large number of generating functions for other polynomial systems also. For instance, we have the following special cases:

(4-A) *Jacobi polynomials*: Applying (4.1) to (1.6), we obtain the following generating relation for Jacobi polynomials proved earlier by Manocha and Sharma [4]

$$
\begin{aligned}
(4.3)\quad & \sum_{n=0}^{\infty}\binom{m+n}{n}P_{m+n}^{(\alpha-n,\beta-n)}(x)t^n \\
&=\left\{1+\frac{1}{2}(x+1)t\right\}^{\alpha}\left\{1+\frac{1}{2}(x-1)t\right\}^{\beta}\cdot P_m^{(\alpha,\beta)}\left(x+\frac{1}{2}(x^2-1)t\right).
\end{aligned}
$$

(4-B) *Extended Jacobi Polynomials.* (4.1) when applied to (1.7) yields the main generating relation of Patil and Thakare [5]

$$
\begin{aligned}
(4.4)\quad & \sum_{n=0}^{\infty}\binom{m+n}{n}F_{m+n}(\alpha-n,\ \beta-n;\ x)t^n \\
&=\left[1-\frac{\lambda t(b-x)}{b-a}\right]^{\alpha}\left[1+\frac{\lambda t(x-a)}{b-a}\right]^{\beta}F_m\left(\alpha,\ \beta;\ x-\frac{\lambda t(x-a)(b-x)}{b-a}\right).
\end{aligned}
$$

(4-*C*) *Srivastava and Singhal Generalized Polynomials.* (4.1) and (1.8) given us the following generating relation for $T_n^{(\alpha-n,\beta-n)}(x,\ a,\ b,\ c,\ d,\ p,\ r)$ determined by Srivastava and Singhal [8].

$$
\begin{aligned}
& \sum_{n=0}^{\infty}\binom{m+n}{n}T_{m+n}^{(\alpha-n,\beta-n)}(x,\ a,\ b,\ c,\ d,\ p,\ r)t^n \\
(4.5)\quad &=\{1+at(cx+d)\}^{\alpha}\{1+ct(ax+b)\}^{\beta}\cdot\exp\{px^r-p(x+t(ax+b)(cx+d))^r\} \\
&\quad\times T_m^{(\alpha,\beta)}(x+t(ax+b)(cx+d),\ a,\ b,\ c,\ d,\ p,\ r)\ .
\end{aligned}
$$

5. Bilateral generating function

In this section we shall prove a theorem on bilateral generating function for $S_n^{(\alpha-n,\beta-n)}(x,\ a,\ b,\ c,\ d,\ e,\ f,\ p,\ r)$. And as a corollary to the theorem we obtain a bilateral generating relation for $T_n^{(\alpha-n,\beta-n)}(x,\ a,\ b,\ c,\ d,\ p,\ r)$ obtained by Srivastava and Singhal [8].

THEOREM. *If*

$$
(5.1)\quad F[x,\ t]=\sum_{n=0}^{\infty}a_nS_n^{(\alpha-n,\beta-n)}(x,\ a,\ b,\ c,\ d,\ e,\ f,\ p,\ r)t^n\ ,
$$

where the a_n *are arbitrary constants, then*

$$
\left[\frac{a\{x+(ax^e+b)(cx^f+d)t\}^e+b}{ax^e+b}\right]^{\alpha}\left[\frac{c\{x+(ax^e+b)(cx^f+d)t\}^f+d}{cx^f+d}\right]^{\beta}
$$

(5.2)
$$\times \exp\{px^r - p(x+(ax^e+b)(cx^f+d)t)^r\} \cdot F\left[x+(ax^e+b)(cx^f+d)t, \frac{yt}{\left(\frac{a\{x+(ax^e+b)(cx^f+d)t\}^e+b}{ax^e+b}\right)\left(\frac{c\{x+(ax^e+b)(cx^f+d)t\}^f+d}{cx^f+d}\right)}\right]$$
$$= \sum_{n=0}^{\infty} S_n^{(\alpha-n,\beta-n)}(x, a, b, c, d, e, f, p, r)\sigma_n(y)t^n ,$$

where $\sigma_n(y)$ is a polynomial of degree n in y given by

(5.3)
$$\sigma_n(y) = \sum_{k=0}^{n} \binom{n}{k} a_k y^k .$$

Proof. We substitute the series expansion of $\sigma_n(y)$ given by (5.3) on the right-hand-side of (5.2) and get

$$\sum_{n=0}^{\infty} S_n^{(\alpha-n,\beta-n)}(x, a, b, c, d, e, f, p, r)\sigma_n(y)t^n$$
$$\sum_{k=0}^{\infty} a_k y^k t^k \sum_{n=0}^{\infty} \binom{n+k}{k} S_{n+k}^{(\alpha-n-k,\beta-n-k)}(x, a, b, c, d, e, f, p, r)t^n .$$

It follows therefore from (4.1) of the previous section that

$$\sum_{n=0}^{\infty} S_n^{(\alpha-n,\beta-n)}(x, a, b, c, d, e, f, p, r)\sigma_n(y)t^n$$
$$= \left[\frac{a\{x+(ax^e+b)(cx^f+d)^e+b}{ax^e+b}\right]^\alpha \left[\frac{c\{x+(ax^e+b)(cx^f+d)t\}^f+d}{cx^f+d}\right]^\beta$$
$$\times \exp\{px^r - p(x+(ax^e+b)(cx^f+d)t)^r\}$$
$$\times \sum_{k=0}^{\infty} a_k \left\{\frac{yt}{\left(\frac{a\{x+(ax^e+b)(cx^f+d)t\}^e+b}{ax^e+b}\right)\left(\frac{c\{x+(ax^e+b)(cx^f+d)t\}^f+d}{cx^f+d}\right)}\right\}^k$$
$$\times S_k^{(\alpha-k,\beta-k)}(x+(ax^e+b)(cx^f+d)t, a, b, c, d, e, f, p, r) ,$$

which in view of (5.1) proves the the theorem. Alternatively, the above theorem may be deduced by applying the general result on bilateral generating functions given in [7] to (4.1). Applying the above theorem to (1.8) we obtain the following particular case of our theorem.

COROLLARY. *If*

(5.4)
$$F[x, t] = \sum_{n=0}^{\infty} a_n T_n^{(\alpha-n,\beta-n)}(x, a, b, c, d, p, r)t^n ,$$

where the a_n are arbitrary coefficients. Then

(5.6)
$$\{1+at(x+d)\}^\alpha \{1+ct(ax+b)\}^\beta \cdot \exp\{px^r - p(x+t(ax+b)(cx+d))^r\}$$
$$\times F\left[x+t(ax+b)(cx+d), \frac{yt}{\{1+at(cx+d)\}\{1+ct(ax+b)\}}\right]$$

$$=\sum_{n=0}^{\infty} T_n^{(\alpha-n,\beta-n)}(x,\ a,\ b,\ c,\ d,\ p,\ r)\sigma_n(y)t^n\ ,$$

where $\sigma_n(y)$ *is given by*

$$\sigma_n(y)=\sum_{k=0}^{n}\binom{n}{k}a_k y^k\ . \tag{5.7}$$

6. Trilateral generating function

In this section we prove the following theorem.

THEOREM. *If*

$$F[x,\ y,\ t]=\sum_{n=0}^{\infty} a_n S_n^{(\alpha-n,\beta-n)}(x,\ a,\ b,\ c,\ d,\ e,\ f,\ p,\ r)g_n(y)t^n \tag{6.1}$$

be a bilateral generating relation, where $a_n\neq 0$ *are arbitrary constants and the* $g_n(y)$ *are arbitrary classical polynomials or functions, then the following relation holds*:

$$\begin{aligned}\sum_{n=0}^{\infty} & S_n^{(\alpha-n,\beta-n)}(x,\ a,\ b,\ c,\ d,\ e,\ f,\ p,\ r)\cdot\sigma_n(y,\ z)t^n\\ &=\left[\frac{a\{x+(ax^e+b)(cx^f+d)t\}^e+b}{ax^e+b}\right]^{\alpha}\left[\frac{c\{x+(ax^e+b)(cx^f+d)t\}^f+d}{cx^f+d}\right]^{\beta}\\ &\quad\times\exp\{px^r-p(x+(ax^e+b)(cx^f+d)t)^r\}F\Bigg[x+(ax^e+b)(cx^f+d)t,\ y\ ,\\ &\quad\times\frac{zt}{\left(\dfrac{a\{x+(ax^e+b)(cx^f+d)t\}^e+b}{ax^e+b}\right)\left(\dfrac{c\{x+(ax^e+b)(cx^f+d)t\}^f+d}{cx^f+d}\right)}\Bigg]\ ,\end{aligned} \tag{6.2}$$

where

$$\sigma_n(y,\ z)=\sum_{k=0}^{n}\binom{n}{k}a_k g_k(y)z^k\ . \tag{6.3}$$

Proof. We substitute the series expansion of $\sigma_n(y,\ z)$ given by (6.3) on the left-hand-side of (6.2) and get

$$\begin{aligned}\sum_{n=0}^{\infty} & S_n^{(\alpha-n,\beta-n)}(x,\ a,\ b,\ c,\ d,\ e,\ f,\ p,\ r)\cdot\sigma_n(y,\ z)t^n\\ &=\sum_{k=0}^{\infty} a_k(zt)^k g_k(y)\sum_{n=0}^{\infty}\binom{n+k}{k}S_{n+k}^{(\alpha-n-k,\beta-n-k)}(x,\ a,\ b,\ c,\ d,\ e,\ f,\ p,\ r)t^n\ .\end{aligned}$$

On summing the inner series with the help of (4.1) and then interpreting the resulting expression by means of (6.1) we are immediately led to the the theorem.

On applying the above theorem to (1.8), we obtain the following trilateral generating relation which is believed to be new for Srivastava's generalized polynomials.

COROLLARY. *If*

$$F[x, y, t]=\sum_{n=0}^{\infty} a_n T_n^{(\alpha-n,\beta-n)}(x, a, b, c, d, e, p, r)g_n(y)t^n \tag{6.4}$$

be a bilateral generating relation, where $a_n \neq 0$ *are arbitrary constants and the* $g_n(y)$ *are arbitrary classical polynomials or functions, then the following generating relation holds*

$$\begin{aligned}&\sum_{n=0}^{\infty} T_n^{(\alpha-n,\beta-n)}(x, a, b, c, d, p, r)\sigma_n(y, z)t^n \\ &=\{1+at(cx+d)\}^{\alpha}\{1+ct(ax+b)\}^{\beta}\cdot\exp\{px^r-p(x+t(ax+b)(cx+d))^r\} \\ &\quad\times F\left[x+t(ax+b)(cx+d),\ y,\ \frac{zt}{\{1+at(cx+d)\}\{1+ct(ax+b)\}}\right],\end{aligned} \tag{6.5}$$

where

$$\sigma_n(y, z)=\sum_{k=0}^{n}\binom{n}{k}a_k g_k(y)z^k\ . \tag{6.6}$$

Refereces

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